JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **218,** 313–324 (1998)
ARTICLE NO. AY975751

NOTE

Note on Critical Exponents for a System of Heat Equations Coupled in the Boundary Conditions*

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Submitted by Konstantin A. Lurie

Received December 2, 1996

This note establishes the blow up estimates near the blow up time for a system of heat equations coupled in the boundary conditions. Under certain assumptions, the exact rate of blow up is established. We also prove that the only solution with vanishing initial values when $pq \geq 1$ is the trivial one.

1. INTRODUCTION

In this note we study the blow up estimates for a system of heat equations coupled in the boundary conditions, namely

$$\begin{cases} u_t = u_{xx}, v_t = v_{xx}, & x > 0, t > 0, \\ -\dfrac{\partial u}{\partial x} = v^p, -\dfrac{\partial v}{\partial x} = u^q, & x = 0, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x > 0, \end{cases} \tag{1}$$

* Project supported by the National Natural Science Foundation of China

0022-247X/98 $25.00

where $pq > 1$ and both $u_0(x)$ and $v_0(x)$ are nonnegative bounded functions.

There are many results for the half space problem (see [2, 5, 9]). Galaktionov and Levine [9] considered the boundary value problem

$$\begin{cases} u_t = u_{xx}, & x > 0, t > 0, \\ -u_x = u^p, & x = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x > 0, \\ -u_{0x}(0) = u_0^p(0). \end{cases} \tag{2}$$

They showed that if $1 < p \leq 2$, then $u(x,t)$ blows up in a finite time for all nontrivial u_0; whereas if $p > 2$, then $u(x,t)$ becomes unbounded for large u_0 and $u(x,t)$ exists globally for small initial data. Their results extend to the half space problem (1) in [2]. It is shown in [2] that there exists a Fujita type critical exponent for the system (1). Questions about blow up rate, blow up set, asymptotic behavior, and uniqueness have been studied by a number of authors. There is a nice survey [5] on this subject.

In the one-space-dimensional case, blow up rate and blow up set were established in [2] under certain assumptions on the initial data and the condition $\min\{p,q\} > 1$. When $pq < 1$, a nontrivial solution with vanishing initial values was also constructed. Their results are expected if one compares the system (1) to the equation $u_t = \Delta u + u^p$ with zero Dirichlet boundary condition or Cauchy problem (see [10, 11]).

In this note we shall establish the blow up rate estimates for the case $pq > 1$ and remove the condition $\min\{p,q\} > 1$ and certain assumptions on the initial data in [2]. We shall also prove a uniqueness result of (1) with vanishing initial values when $pq > 1$.

Our main results read as follows:

THEOREM 1. *Assume that T is blow up time and that $u_0, v_0 \in C^2 \cap L^\infty(0, +\infty)$ satisfy*

$$(-1)^{(i)} u_0^{(i)}(x) \geq 0, \qquad (-1)^{(i)} v_0^{(i)}(x) \geq 0, i = 0,1,2, \tag{3}$$

$$-u_{0x}(0) \leq v_0^p(0), \qquad -v_{0x}(0) \leq u_0^q(0). \tag{4}$$

Then there exist positive constants C_i, $i = 1,2,3,4$, such that

$$C_1(T-t)^{-\tau_1} \leq u(0,t) \leq C_2(T-t)^{-\tau_1}$$

and

$$C_3(T-t)^{-\tau_2} \leq v(0,t) \leq C_4(T-t)^{-\tau_2},$$

where $\tau_1 = (p+1)/2(pq-1)$ and $\tau_2 = (q+1)/2(pq-1)$.

Under the assumptions on initial values of Theorem 3.4 in [2], we can remove the condition $\min\{p, q\} > 1$ and simplify their proof.

THEOREM 2. *Assume that* $pq > 1$. *Assume that* T *is a blow up time and that* $u_0, v_0 \in C^3 \cap L^\infty(0, +\infty)$ *satisfy*

$$-u_0'(0) = v_0^p(0), \qquad -v_0'(0) = u_0^q(0) \tag{5}$$

$$-u_0'''(0) = pv^{p-1}(0)v_0''(0), \qquad -v_0'''(0) = qu^{q-1}(0)u_0''(0), \tag{6}$$

$$(-1)^{(i)}u_0^{(i)}(x) \geq 0, \qquad (-1)^{(i)}v_0^{(i)}(x) \geq 0, i = 0, 1, 2, 3, \tag{7}$$

$$\lim_{x \to +\infty} u_0(x) = 0, \qquad \lim_{x \to +\infty} v_0(x) = 0, \tag{8}$$

$$u_{0x}(x) \leq -v_0^p(x), \qquad v_{0x}(x) \leq -u_0^q(x). \tag{9}$$

Then there exists a positive constant C_i, $i = 1, 2, 3, 4$, *such that*

$$C_1(T-t)^{-\tau_1} \leq u(0,t) \leq C_2(T-t)^{-\tau_1}$$

and

$$C_3(T-t)^{-\tau_2} \leq v(0,t) \leq C_4(T-t)^{-\tau_2},$$

where $\tau_1 = (p+1)/2(pq-1)$ *and* $\tau_2 = (q+1)/2(pq-1)$.

THEOREM 3. *Assume that* $pq \geq 1$. *Then the only solution of the problem*

$$\begin{cases} u_t = \Delta u, v_t = \Delta v, & x \in R_+^N, t > 0, \\ -\dfrac{\partial u}{\partial x_1} = v^p, -\dfrac{\partial v}{\partial x_1} = u^q, & x_1 = 0, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in R_+^N \end{cases} \tag{10}$$

with $u_0 = v_0 = 0$ *is the trivial one, i.e.,* $u = v = 0$.

For definiteness, we may always assume $q \geq p$ throughout the paper.

2. THE PROOFS OF THE THEOREMS

Throughout this paper, we shall use C, C_i, c, and c_i to denote various generic constants if there is no confusion.

LEMMA 1. *Assume that* (u, v) *is the solution of* (10).

(i) *If* $q \geq p$, *then there exists a positive constant* C_5 *such that*

$$v \geq C_5 u^{(q+1)/(p+1)}; \tag{11}$$

(ii) *If $p \geq q$, then there exists a positive constant C_6 such that*

$$u \geq C_6 v^{(p+1)/(q+1)}.$$

Proof. (i) Set $J = b_1 v^{p+1} - b_2 u^{q+1}$, where $b_1, b_2 > 0$ satisfy

$$b_1(p+1) - b_2(q+1) \geq 0$$

and

$$b_1 v_0^{p+1}(x) - b_2 u_0^{q+1}(x) \geq 0.$$

Then it can be verified that J satisfies

$$J_t - \Delta J + \mathbf{a}(u, v, \nabla J) \cdot \nabla J + b(u,v) J = (q-p) F(u,v),$$

where

$$\mathbf{a}(u, v, \nabla J) = \frac{p\left(\nabla J + 2b_2(q+1)u^q \nabla u\right)}{b_1(p+1)v^{p+1}},$$

$$b(u,v) = -\frac{b_2 p(q+1)^2 u^{q-1} |\nabla u|^2}{b_1(p+1)v^{p+1}},$$

$$F(u,v) = \frac{b_2(q+1)}{p+1} u^{q-1} |\nabla u|^2 \geq 0,$$

$$-\frac{\partial J}{\partial x_1} = -\left(b_1(p+1)v^p \frac{\partial v}{\partial x_1} - b_2(q+1)u^q \frac{\partial u}{\partial x_1}\right)$$

$$= b_1(p+1)v^p u^q - b_2(q+1)u^q v^p$$

$$= u^q v^p \big(b_1(p+1) - b_2(q+1)\big) \geq 0, \qquad x_1 = 0,$$

$$J(x,0) = b_1 v_0^{p+1}(x) - b_2 u_0^{q+1}(x) \geq 0.$$

Therefore, we have

$$J \geq 0,$$

i.e.,

$$v \geq C_5 u^{(q+1)/(p+1)}.$$

(ii) The proof is similar to that of (i). ∎

Proof of Theorem 1. By the maximum principle, it follows from (3)–(4) that

$$u, u_t, v, v_t \geq 0 \qquad \text{and} \qquad u_x, v_x \leq 0 \text{ for } t \in (0,T).$$

Hence $u(0,t) = \max_{x \geq 0} u(x,t)$ and $v(0,t) = \max_{x \geq 0} v(x,t)$. Recall that the Green's function $G(x;y;t)$ for the heat equation in R_+^N satisfying $\partial G/\partial y = 0$ at $y = 0$ is given by

$$G(x;y;t) = (4\pi t)^{-1/2}\left(\exp\left(-\frac{(x-y)^2}{4t}\right) + \exp\left(-\frac{(x+y)^2}{4t}\right)\right).$$

We have the representation formulae [2] for the solution of (1):

$$u(x,t) = \int_0^{+\infty} G(x;y;t)u_0(y)\,dy + \int_0^t G(x;0;t-s)v^p(0,s)\,ds \tag{12}$$

and

$$v(x,t) = \int_0^{+\infty} G(x;y;t)v_0(y)\,dy + \int_0^t G(x;0;t-s)u^q(0,s)\,ds. \tag{13}$$

These are the so-called *variation of canstants formulae* (cf. [1, 13]).

First we derive the blow up rate estimates from above.

LEMMA 2. *Under the assumptions of Theorem* 1, *there exist positive constants* C_2 *and* C_4 *such that*

$$u(0,t) \leq C_2(T-t)^{-\tau_1}, \tag{14}$$

$$v(0,t) \leq C_4(T-t)^{-\tau_2}, \tag{15}$$

where $\tau_1 = (p+1)/2(pq-1)$ *and* $\tau_2 = (q+1)/2(pq-1)$.

Proof. By (12) and (11) we have, for $0 < t < T$,

$$\begin{aligned} u(0,t) &\geq \int_0^t (\pi(t-s))^{-1/2} v^p(0,s)\,ds \\ &\geq \int_0^t (\pi(T-s)^{-1/2} C_5^p u^{p(q+1)/(p+1)}(0,s)\,ds = c_1 I(t). \end{aligned}$$

As in [12], it follows that

$$\begin{aligned} I'(t) &= (T-t)^{-1/2} u^{p(q+1)/(p+1)}(0,t) \\ &\geq c_1^{p(q+1)/(p+1)}(T-t)^{-1/2} I^{p(q+1)/(p+1)}(t). \end{aligned}$$

Integrating the above inequality we obtain (here we can assume $I(T) = +\infty$)

$$I(t) \leq C^*(T-t)^{-(p+1)/[2(pq-1)]}. \tag{16}$$

On the other hand, for any $T/2 \le t < T$ and $0 < \bar{t} < t$,

$$I(t) \ge \int_{\bar{t}}^{t} \frac{u^{p(q+1)/(p+1)}(0,s)}{\sqrt{T-s}}\,ds$$

$$\ge u^{p(q+1)/(p+1)}(0,\bar{t}) \int_{\bar{t}}^{t} \frac{ds}{\sqrt{T-s}}$$

$$\ge u^{p(q+1)/(p+1)}(0,\bar{t})\,\frac{t-\bar{t}}{\sqrt{T-\bar{t}}}. \tag{17}$$

Combining (16) and (17), we get, for $T/2 \le t < T$ and $0 < \bar{t} < t$,

$$u^{p(q+1)/(p+1)}(0,\bar{t})\,\frac{t-\bar{t}}{\sqrt{T-\bar{t}}} \le C^*(T-t)^{-(p+1)/[2(pq-1)]}.$$

Taking $t = (\bar{t} + T)/2$ in above inequality, we have

$$u^{p(q+1)/(p+1)}(0,\bar{t}) \le c_2(T-\bar{t})^{-(p+1)/2(pq-1)-1/2},$$

i.e.,

$$u(0,\bar{t}) \le C_2(T-\bar{t})^{-(p+1)/[2(pq-1)]}.$$

This implies that (14) holds.

If (15) fails, then there is an increasing sequence of times $t_n \to T^-$ such that

$$v(0,t_n) \ge c_n(T-t_n)^{-\tau_2} \tag{18}$$

and $c_n \to +\infty$ as $n \to +\infty$ (see [10, 11]).

Since $v_t \ge 0$, by (12) and (18) we get for any $t_n \le t < T$

$$u(0,t) \ge \int_{t_n}^{t} \pi^{-1/2}\,\frac{ds}{\sqrt{T-s}}\,v^p(0,t_n)$$

$$\ge c_n^p \pi^{-1/2}\,\frac{t-t_n}{\sqrt{T-t_n}}\,(T-t_n)^{-p(q+1)/2(pq-1)}. \tag{19}$$

By (14) and (19) we get, for any $t_n \le t < T$,

$$c_n^p \pi^{-1/2}\,\frac{t-t_n}{\sqrt{T-t_n}}\,(T-t_n)^{-p(q+1)/2(pq-1)} \le C_2(T-t)^{-(p+1)/2(pq-1)}.$$

Taking $t = (t_n + T)/2$ in the above equality, we get

$$\frac{1}{2} c_n^p \pi^{-1/2} \le C_2 2^{(p+1)/2(pq-1)}.$$

There is a contradiction as $n \to +\infty$ because $c_n \to +\infty$ as $n \to +\infty$ and hence (15) holds. ∎

We shall derive the estimate from below by using the idea of [12].

LEMMA 3. *Under the assumptions of Theorem* 1, *there exist positive constants C_1 and C_3 such that*

$$u(0,t) \ge C_1 (T-t)^{-\tau_1}, \tag{20}$$

$$v(0,t) \ge C_3 (T-t)^{-\tau_2}, \tag{21}$$

where $\tau_1 = (p+1)/2(pq-1)$ *and* $\tau_2 = (q+1)/2(pq-1)$.

Proof. For any $0 \le z < t < T$ and $x \ge 0$, we have Green's identity [1]

$$u(x,t) = \int_0^{+\infty} G(x,y,t) u(y,z)\, dy + \int_z^t G(x;0;t-s) v^p(0,s)\, ds$$

and

$$v(x,t) = \int_0^{+\infty} G(x,y,t) v(y,z)\, dy + \int_z^t G(x;0;t-s) u^q(0,s)\, ds.$$

Since $\|u(\cdot,t)\|_\infty = u(0,t)$, $\|v(\cdot,t)\|_\infty = v(0,t)$, $u_t \ge 0$, and $v_t \ge 0$, we have

$$u(0,t) \le u(0,z) + \int_z^t (\pi(t-s))^{-1/2}\, ds \cdot v^p(0,t)$$

and

$$v(0,t) \le v(0,z) + \int_z^t (\pi(t-s))^{-1/2}\, ds \cdot u^q(0,t);$$

i.e.,

$$u(0,t) \le u(0,z) + C_7 \sqrt{T-z}\, v^p(0,t) \tag{22}$$

and

$$v(0,t) \le v(0,z) + C_8 \sqrt{T-z}\, u^q(0,t). \tag{23}$$

By (11) and (23) we get

$$v(0,t) \le v(0,z) + C_8 \sqrt{T-z}\, C_5^{-q(p+1)/(q+1)} v^{q(p+1)/(q+1)}(0,t). \tag{24}$$

By the assumptions, T is the blow up time; therefore

$$\lim_{t\to T^-} v(0,t) = +\infty$$

and hence we can choose $t < T$ such that $v(0,t) = 2v(0,z)$, and the inequality (24) becomes

$$v(0,z) \le C_8\sqrt{T-z}\,C_5^{-q(p+1)/(q+1)}2^{q(p+1)/(q+1)}v^{q(p+1)/(q+1)}(0,z).$$

The above inequality implies that (21) holds.

Since $v_t \ge 0$, by (12) and (21) we get for any $0 < z < t < T$

$$\begin{aligned} u(0,t) &\ge \int_z^t \pi^{-1/2}\frac{ds}{\sqrt{T-s}}v^p(0,z) \\ &\ge C_3^p\pi^{-1/2}\frac{t-z}{\sqrt{T-z}}(T-z)^{-p(q+1)/2(pq-1)}. \end{aligned}$$

Taking $z = 2t - T$ in the above inequality, we get

$$u(0,t) \ge \frac{1}{\sqrt{2}}C_3^p\pi^{-1/2}2^{-(p+1)/2(pq-1)}(T-t)^{-p(q+1)/2(pq-1)};$$

i.e., (20) holds. ∎

Proof of Theorem 2. Since $pq > 1$ and $q \ge p$, we have $q > 1$. By the results of [2, Theorem 3.4] we have

$$\frac{1}{2}v^{2p}(0,t) \le u_t(0,t)u(0,t) \tag{25}$$

and

$$\frac{1}{2}u^{2q}(0,t) \le v_t(0,t)v(0,t). \tag{26}$$

As in [2], we define on $[0,+\infty)\times[0,T)$

$$K(x,t) = v_x(x,t) + u^q(x,t).$$

The assumption (9) ensures that $K(x,0) \le 0$.

A routine calculation yields

$$K_t - K_{xx} = -q(q-1)u^{q-2}u_x^2.$$

Using the boundary conditions and (8), we have [2]

$$K(0,t) = 0, \qquad \lim_{x \to +\infty} K(x,t) = 0.$$

Thus, by the maximum principle $K \leq 0$ in $[0, +\infty) \times [0, T)$. Therefore, for every $t \in (0, T)$, the function $K(\cdot, t)$ attains its maximum at $x = 0$. Consequently $K_x(0,t) \leq 0$. Writing out the inequality, we see that on $(0, T)$

$$v_t(0,t) \leq qu^{q-1}(0,t)\big(-u_x(0,t)\big) = qu^{q-1}(0,t)v^p(0,t). \tag{27}$$

Using (25)–(27), we can obtain Theorem 2. Indeed, as in [2], we easily obtain

$$u(0,t) \leq C_2(T-t)^{-(p+1)/2(pq-1)} \tag{28}$$

and

$$v(0,t) \geq C_3(T-t)^{-(q+1)/2(pq-1)}. \tag{29}$$

By (25) and (29), we have, for any $z < t < T$,

$$\begin{aligned} u^2(0,t) &\geq \int_0^t v^{2p}(0,s)\,ds \geq \int_z^t v^{2p}(0,s)\,ds \\ &\geq (t-z)v^{2p}(0,z) \geq C_3^{2p}(t-z)(T-z)^{-p(q+1)/(pq-1)}. \end{aligned}$$

Taking $z = 2t - T(<t)$ in the above inequality, we can obtain

$$u(0,t) \geq C_1(T-t)^{-(p+1)/2(pq-1)}. \tag{30}$$

Finally, we shall prove that

$$v(0,t) \leq C_4(T-t)^{-(q+1)/2(pq-1)}. \tag{31}$$

Otherwise, there is an increasing sequence of times $t_n \to T^-$ such that

$$v(0,t_n) \geq c_n(T-t_n)^{-\tau_2} \tag{32}$$

and $c_n \to +\infty$ as $n \to +\infty$ (see [10, 11]).

Since $v_t \geq 0$, by (25) and (32) we get, for any $t_n \leq t < T$,

$$\begin{aligned} u^2(0,t) &\geq \int_0^t v^{2p}(0,s)\,ds \geq \int_{t_n}^t v^{2p}(0,s)\,ds \\ &\geq (t-t_n)v^{2p}(0,t_n) \geq c_n^{2p}(t-t_n)(T-t_n)^{-p(q+1)/(pq-1)}. \end{aligned} \tag{33}$$

By (28) and (33) we get

$$c_n^{2p}(t-t_n)(T-t_n)^{-p(q+1)/(pq-1)} \le C_2^2(T-t)^{-(p+1)/(pq-1)}.$$

Taking $t = (t_n + T)/2$ ($< t$) in the above inequality, we obtain

$$\frac{1}{2}c_n^{2p} \le C_2^2 2^{(p+1)/(pq-1)}.$$

There is a contradiction as $n \to +\infty$ because $c_n \to +\infty$ as $n \to +\infty$ and hence (31) holds.

By (28)–(31) we obtain Theorem 2. ∎

To prove Theorem 3, we first prove

LEMMA 4. *Assume that $\tau \ge 1$. Then the only solution of the problem*

$$\begin{cases} w_t = \Delta w, & x \in R_+^N, t > 0, \\ -\dfrac{\partial w}{\partial x_1} = cw^\tau, & x_1 = 0, t > 0, \\ w(x,0) = 0, & x \in R_+^N \end{cases} \tag{34}$$

is the trivial one, i.e., $w = 0$.

Proof. First we have the representation formulae [2] for the solution of (34),

$$w(x_1, x', t) = \int_0^t G(x;0;t-s)S(t-s)w^\tau(0,x',s)\,ds, \tag{35}$$

where

$$S(t) = (4\pi t)^{-(N-1)/2}\exp\left(-\frac{|x'|^2}{4t}\right), \qquad x' \in R^{N-1}.$$

Then

$$\|w(\cdot,t)\|_\infty \le \int_0^t (\pi(t-s))^{-1/2}\|w(\cdot,s)\|_\infty^\tau\,ds$$

$$\le \frac{2\sqrt{t}}{\sqrt{\pi}} \sup_{0\le s\le t}\|w(\cdot,s)\|_\infty^\tau.$$

Set $f(t) = \sup_{0\le s\le t}\|w(\cdot,s)\|_\infty$. Then we have

$$f(t) \le \frac{2\sqrt{t}}{\sqrt{\pi}} f^\tau(t), \qquad t \ge 0.$$

Since $f(0) = 0$ and $\tau \ge 1$, we have $f(t) = 0$, i.e., $w = 0$. ∎

Proof of Theorem 3. By (10) and (11) we see that v satisfies

$$\begin{cases} v_t = \Delta v, & x \in R_+^N, t > 0, \\ -\dfrac{\partial v}{\partial x_1} \leq C_5^{-q(p+1)/(q+1)} v^{q(p+1)/(q+1)}, & x_1 = 0, t > 0, \\ v(x,0) = 0, & x \in R_+^N. \end{cases}$$

Since $pq \geq 1$, i.e., $q(p+1)/(q+1) \geq 1$, by Lemma 4 and the maximum principle we have $v = 0$. Hence $u = 0$. ∎

Remark 1. Theorem 3 complements Theorem 3.5 in [2].

Remark 2. The results in Lemma 1 and Theorem 3 may extend to the system

$$\begin{cases} u_t = \Delta u, v_t = \Delta v, & x \in R_+^N, t > 0, \\ -\dfrac{\partial u}{\partial x_1} = u^\alpha v^p, -\dfrac{\partial v}{\partial x_1} = u^q v^\beta, & x_1 = 0, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in R_+^N. \end{cases}$$

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